THE END OF NUMERICAL ERROR

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Big problems facing computing

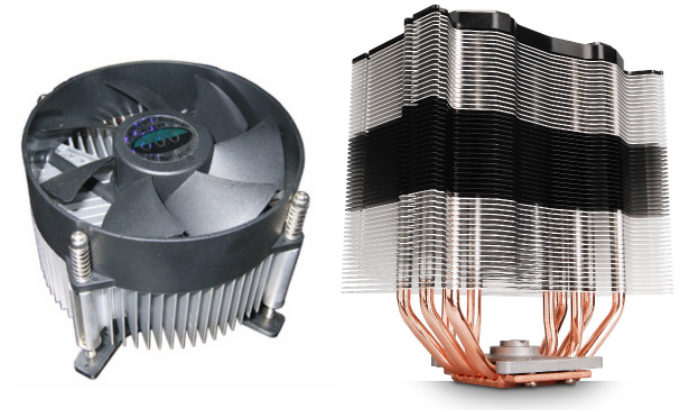
- More hardware parallelism than we know how to use
- Too much energy and power needed per calculation
- Not enough bandwidth (the “memory wall”)
- Rounding errors more treacherous than people realize
- Rounding errors prevent use of multicore methods
- Sampling errors turn physics simulations into guesswork
- Numerical methods are hard to use, require experts
- IEEE floats give different answers on different systems

More parallel hardware than we can use

- Huge clusters usually partitioned into 10s, 100s of cores
- Few algorithms exploit millions of cores except LINPACK
- *Capacity* is not a substitute for *capability*!

Too much power and heat needed

- Huge heat sinks
- 20 MW limit for exascale
- Data center electric bills
- Mobile device battery life
- Heat intensity means *bulk*



Capped title, switched to IDF black.

bandwidth (“ Memory wall”)

	Approximate energy consumed	Approximate time needed
64-bit multiply-add	200 pJ	1 nsec
Read 64 bits from cache	800 pJ	3 nsec
Move 64 bits across chip	2000 pJ	5 nsec
Execute an instruction	7500 pJ	1 nsec
Read 64 bits from DRAM	*12000 pJ*	*70 nsec*

Notice that 12000 pJ @ 3 GHz = 36 watts!

One-size-fits-all overkill 64-bit precision *wastes energy, storage bandwidth*

Floats are worse than people realize

Divide 1 by 3 on calculator:

Multiply by 3, get mistake:

"Computers don't make mistakes"

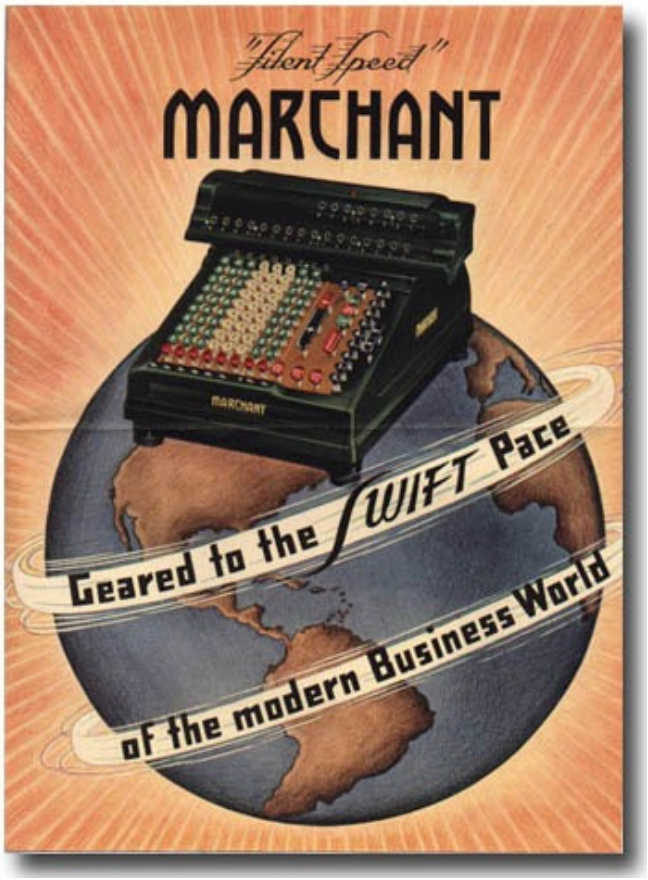
Yes they do. And at *incredibly high speeds.*

Happy 100th Birthday, Floating Point

1914: Torres proposes automatic computing with a fraction and a scaling factor.
2014: We still use a format designed for World War I hardware capabilities!

So what's wrong with floating point?

- Like steampunk; outdated
- OK for *manual* calculations
  - Operator sees, remembers errors
  - Can head off overflow, underflow
- Automatic math *hides* all that
- No one sees processor "flags"
- Disobeys algebraic laws
- Wastes bit patterns as NaNs



Billions of Not-a-Number types…why?

Analogy: Printing in 1970 vs. 2014

1970: 30 sec per page

```
      DISK OPERATING SYSTEM/360 FORTRAN   360N-FO-451   CL
C     ROBERT GLASER, RANDALLSTOWN SENIOR, GROUP A, P AND S
C     PRIME NUMBERS
      DO 100 I=1,1000
      J=2
      K=2
    2 L=J*K
      IF (L-I) 10,100,10
   10 M=2+3
      IF (K-I) 20,3,3
   20 K=K+1
      GO TO 2
    3 K=2
      IF (J-I) 5,4,4
    5 J=J+1
      GO TO 2
    4 WRITE (3,6) I
    6 FORMAT (I10)
  100 CONTINUE
      STOP
      END
```

2013: 30 sec per page

We use faster technology for *better* prints, not for thousands of lousy prints per second. Why not do the same thing with computer arithmetic?

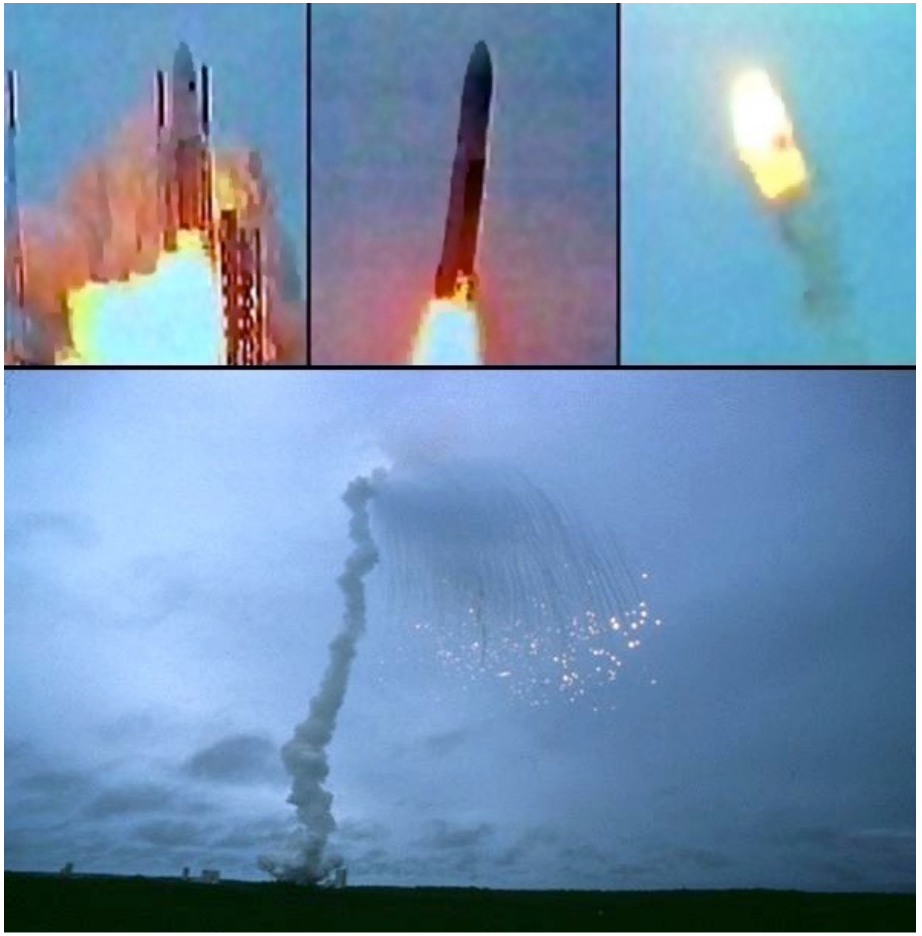
The “Original Sin” of Computer Math

“The computer cannot give you the exact value, sorry. Use *this* value instead. It’s close.”

Float Disaster: The Ariane 5

- $0.7 billion gone in seconds
- 64-bit float measured speed
- 16-bit guidance system; *oops*

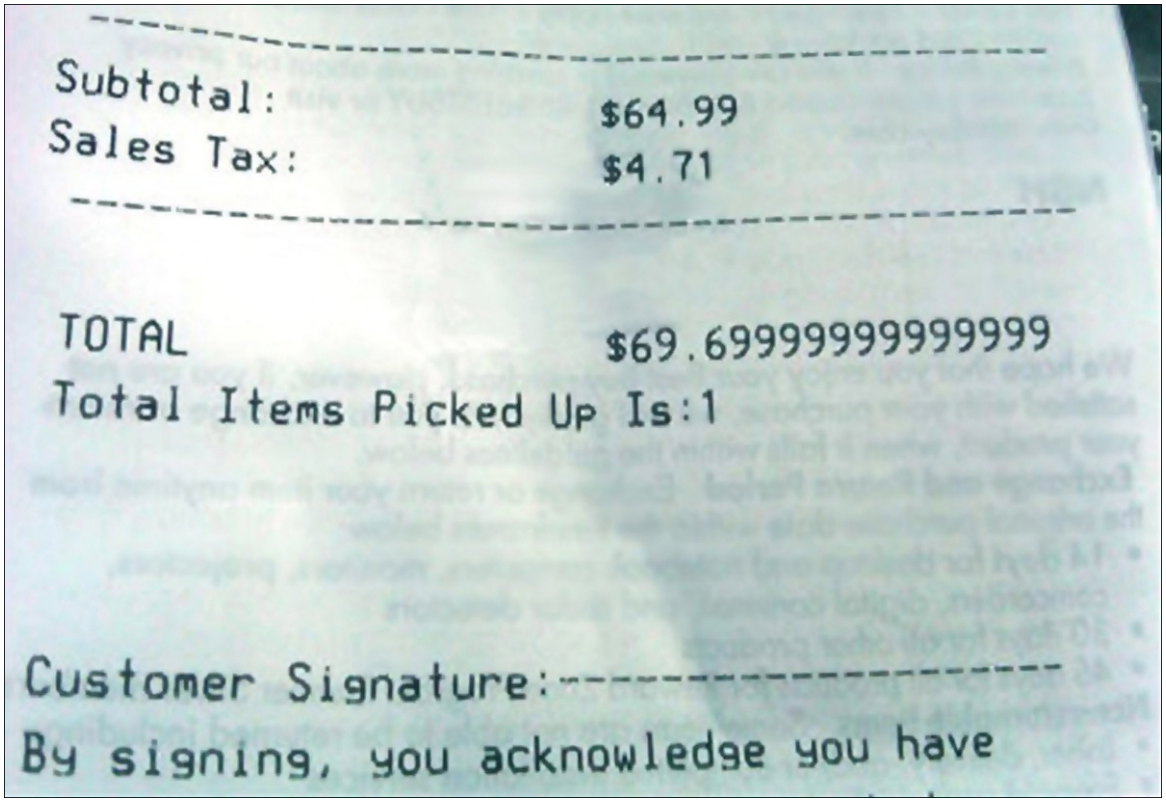
Why do *programmers* have to manage storage sizes when computers are much better at doing it?



The Sleipner Oil Platform Disaster

- Float error in structural analysis; collapsed to ocean floor
- ~$1 billion loss (in 2014 dollars)

Is this hilarious, or just disgusting?



Sampling errors

- Using a single exact value to represent a range of values
- We are still using ancient numerical methods from when switching elements were *expensive*, communication *free*.
- Computing as a “third branch of science” isn’t science if all it produces are *guesses* in physical simulations

Floats *prevent use of multicore*

- No associative property for floats
- $(a + b) + (c + d)$ (parallel) $\neq ((a + b) + c) + d$ (serial)
- Looks like a “wrong answer”
- Programmers trust serial, reject parallel
- Floats indicate rounding, overflow, underflow bits in *processor registers that no one ever sees.*

Some terminology

- Rounding error: substituting an exact number for the correct one
- Sampling error: representing a varying quantity with an exact value
- Overflow: substituting ±∞ for a number too large to store
- Underflow: substituting 0 for a number too small to store
- Precision: Number of bits available to store a number
- Accuracy: Number of correct digits in a result
- ULP: Unit in the Last Place, or Unit of Least Precision

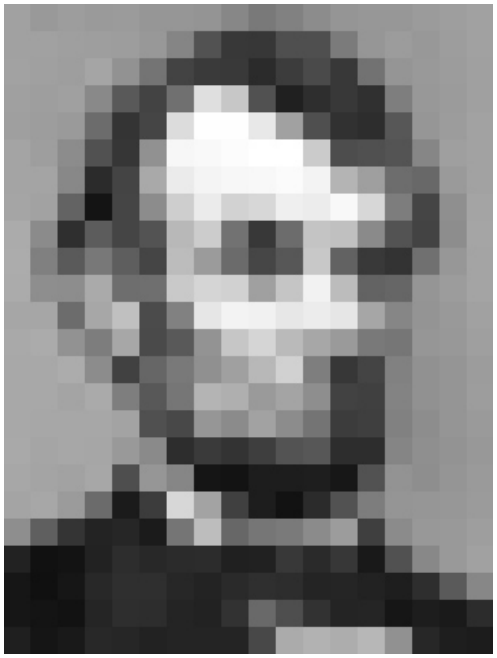
Precision versus Accuracy

150,000 pixels

$\pi \approx$ `3.900000000000001`

High precision.

Low accuracy.



432 pixels

$\pi \approx$ `3.14`

Low precision.

High accuracy.

A New Number Format: The Unum

- Universal numbers
- **Superset** of IEEE types
- Integers➔floats➔unums
- No rounding, no overflow to ∞, no underflow to zero
- They obey algebraic laws!
- Safe to parallelize
- *Fewer bits* than floats
- But… they're *new*
- Some people don't like *new*

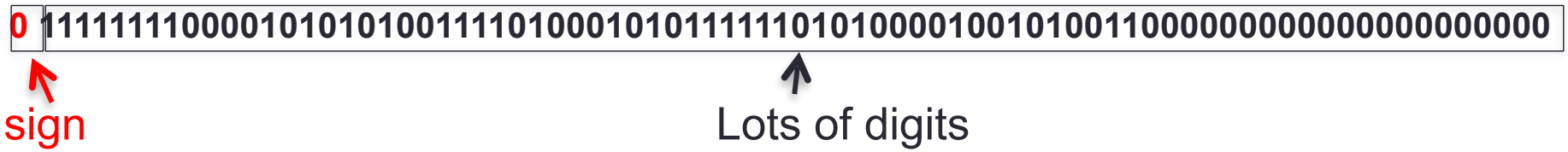
"You can't boil the ocean."

—Former Intel exec, when shown the unum idea

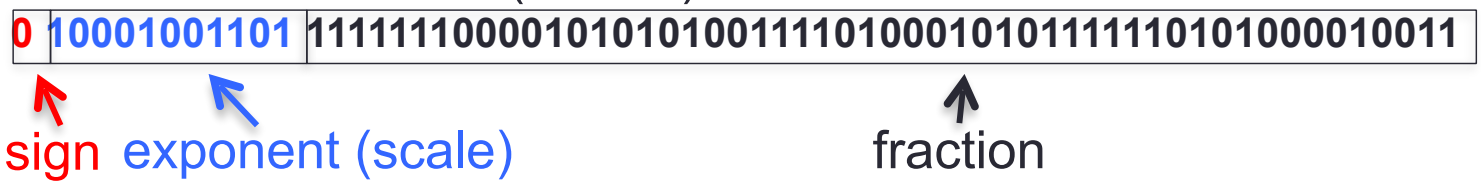
Three ways to express a big number

Avogadro's number: ~6.022×10^{23} atoms or molecules

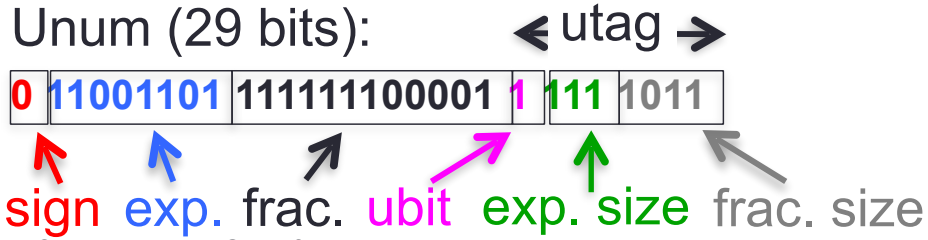
Integer (80 bits):



IEEE Standard Float (64 bits):



Unum (29 bits):



Self-descriptive "utag" bits track and manage uncertainty, exponent size, and fraction size

Why unums use fewer bits than floats

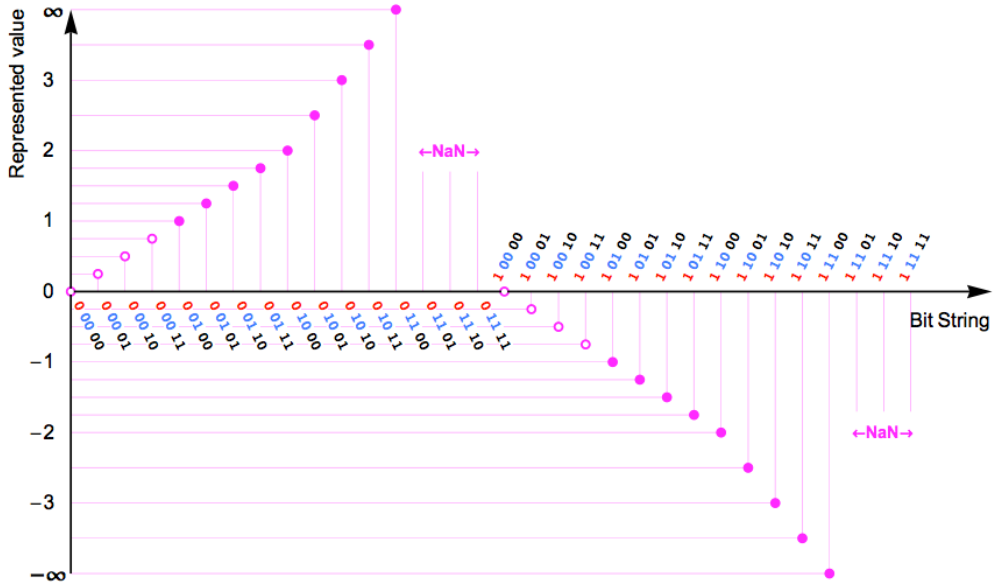
- Exponent smaller by about 5 – 10 bits on average
- Trailing zeroes in fraction compressed away, saves ~2 bits
- Values close to 1 are far more common in real programs
- Cancellation removes bits and the need to store them

IEEE Standard Float (64 bits):

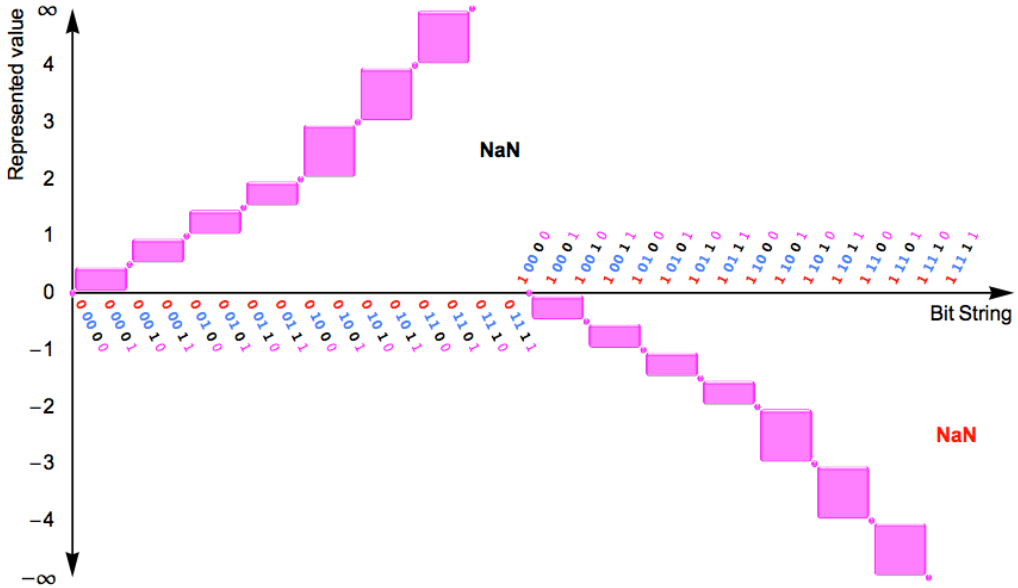
Unum (29 bits):

Open *ranges*, not just exact points

Bit string meanings
using IEEE Float rules



Bit string meanings
in unum format



Complete representation of *all* real numbers using a finite number of bits

Breakthroughs enabled by unums

- Bit-identical results across systems, unlike IEEE floats
- First error-free polynomial evaluator
- Solution to “The Table-Makers Dilemma”
- First deterministic evaluation of x^y
- First massively parallel, bounded ODE solver
- Universal solver for $f(x) = 0$ when f is continuous
- Provably bounded physics simulations – real science!
- P = NP for finding maxima, minima, roots perfectly

The Warlpiri unums

Before the aboriginal Warlpiri of Northern Australia had contact with other civilizations, their counting system was "One, two, many."

Maybe they were onto something.

Floating Point II: The Wrath of Kahan

- Berkeley professor William Kahan is the father of modern IEEE Standard floats
- Also the authority on their many dangers
- Every idea to fix floats faces his tests that expose how new idea is *even worse*

Working unum environment completed August 13, 2013.

Can unums survive the wrath of Kahan?

A Typical Kahan Challenge

"Define functions with: $E(0) = 1,\ E(z) = \frac{e^z - 1}{z}.\ Q[x] = \left| x - \sqrt{x^2 + 1} \right| - \frac{1}{x + \sqrt{x^2 + 1}}.\ H(x) = E(Q(x))^2).$

Compute $H(x)$ for x = 15.0, 16.0, 17.0, 9999.0. Repeat with more precision, say using BigDecimal."

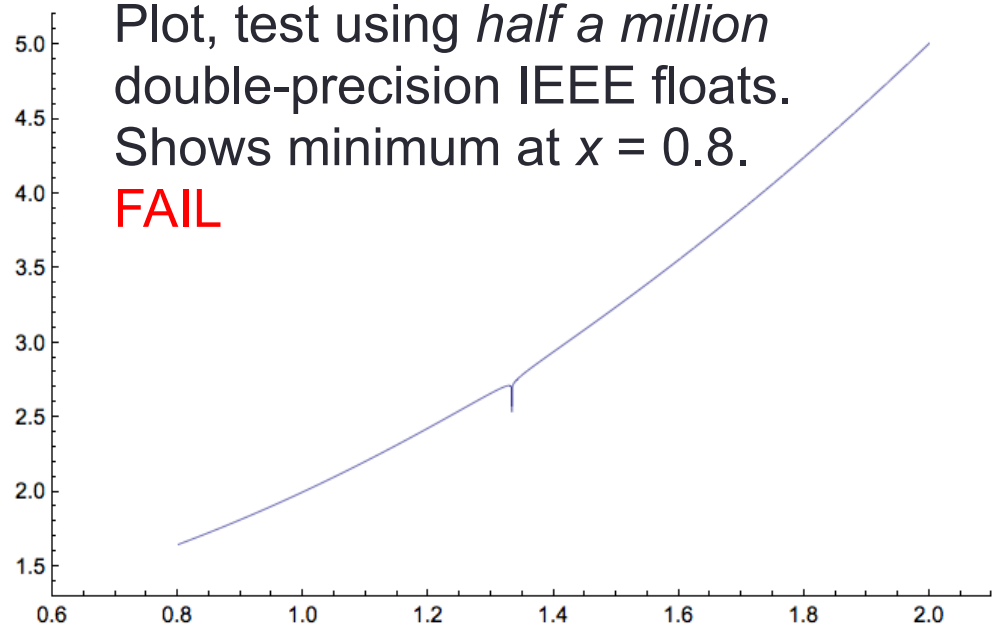
- Correct answer: (1, 1, 1, 1).
- IEEE 32-bit: (0, 0, 0, 0) **FAIL**
- IEEE 64-bit: (0, 0, 0, 0) **FAIL**
- Myth: "Getting the same answer with increased precision means the answer is correct."
- IEEE 128-bit: (0, 0, 0, 0) **FAIL**
- Extended precision math packages: (0, 0, 0, 0) **FAIL**
- Interval arithmetic: Um, somewhere between –∞ and ∞. **EPIC FAIL**
- Unums, ***6-bit*** average size: (1, 1, 1, 1) **CORRECT**

I have been unable to find a problem that "breaks" unum math.

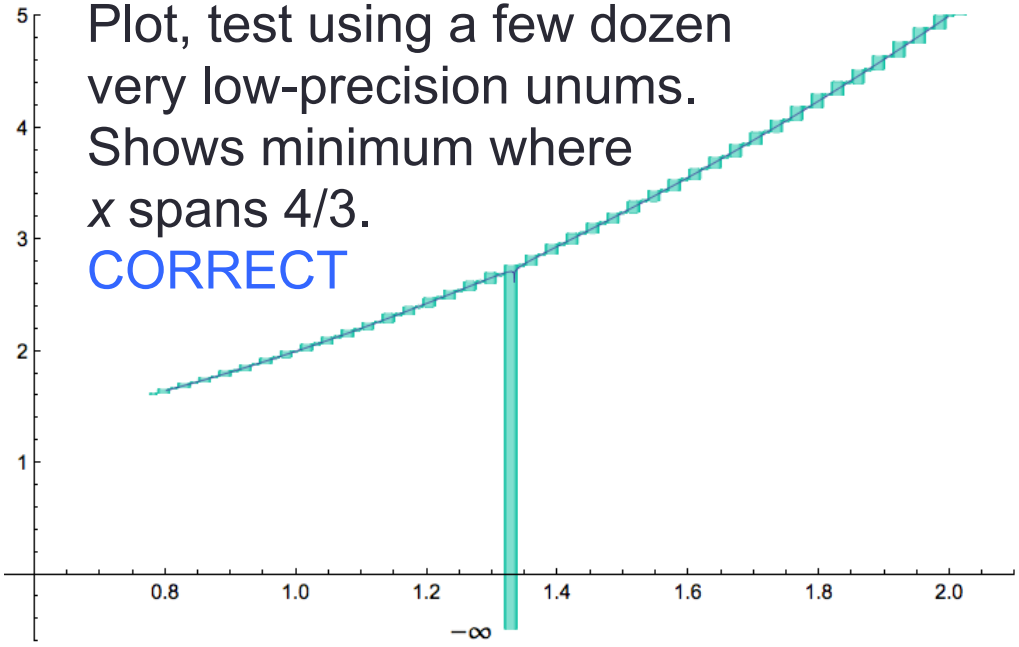
Kahan's "Smooth Surprise"

Find minimum of $\log(|3(1-x)+1|)/80 + x^2 + 1$ in $0.8 \le x \le 2.0$

Plot, test using *half a million* double-precision IEEE floats.
Shows minimum at $x = 0.8$.
FAIL



Plot, test using a few dozen very low-precision unums.
Shows minimum where x spans 4/3.
CORRECT



Kahan on the computation of powers

"Nobody knows how much it would cost to compute y^w correctly rounded for every two floating-point arguments at which it does not over/underflow... No general way exists to predict how many extra digits will have to be carried to compute a transcendental expression and round it correctly to some preassigned number of digits. Even the fact (if true) that a finite number of extra digits will ultimately suffice may be a deep theorem." —*William Kahan*

I sent Kahan my fixed-cost, fixed-storage method, and he said it looked "impractical."

I asked if he had a method that shows the following computation is *exact:*

$$5.9604644775390625^{0.875} = 4.76837158203125$$

Have not heard from him since.

Rump's Royal Pain

Compute $333.75y^6 + x^2(11x^2y^2 - y^6 - 121y^4 - 2) + 5.5y^8 + x/(2y)$
where $x = 77617$, $y = 33096$.

- Looks straightforward; try it with a calculator sometime
- Using IBM (pre-IEEE Standard) floats, Rump got
  - 1.172603 in 32-bit precision
  - 1.1726039400531 in 64-bit precision
  - 1.17260394005317**8** in 128-bit precision
- Using IEEE double precision: 1.18059×10^{21}
- **Correct answer: –1.4411518807585587⋯!** Didn't even get *sign* right

With unums: **Correct answer** to 23 decimals using an average of only ***75*** bits per number. Not even IEEE 128-bit precision can do that.

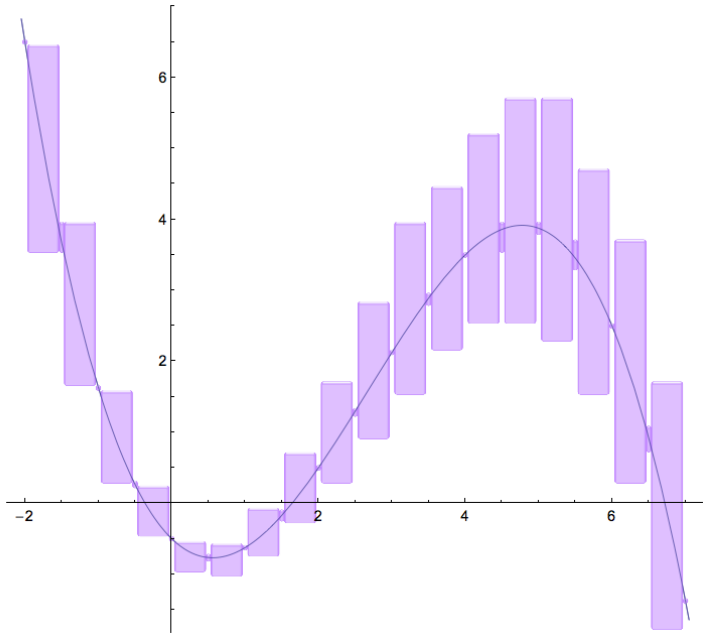
Fundamental principle

Bound the answer as tightly as possible within the numerical environment, or admit defeat.

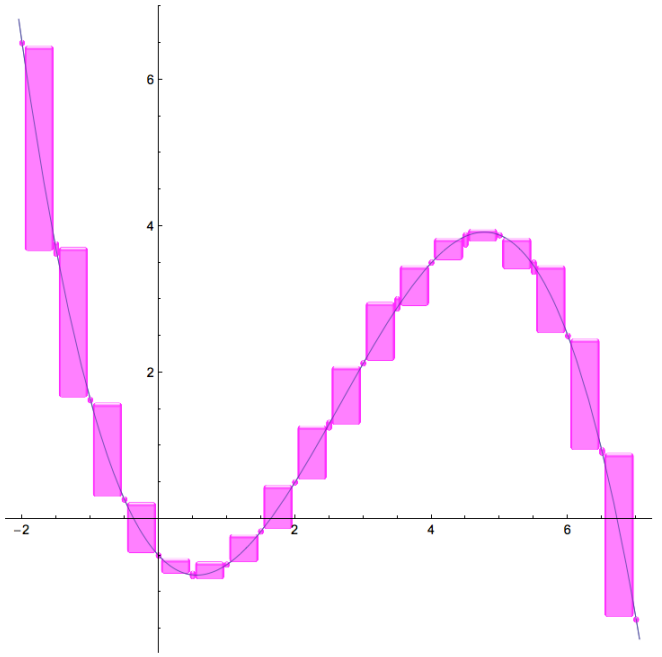
- No more guessing
- No more “the error is $O(h^n)$” type estimates
- The smaller the bound, the greater the knowledge
- Performance is *knowledge per second*

Polynomial evaluation solved at last

Mathematicians have sought this for at least 60 years.



"Dependency Problem" creates sloppy range when input is an interval



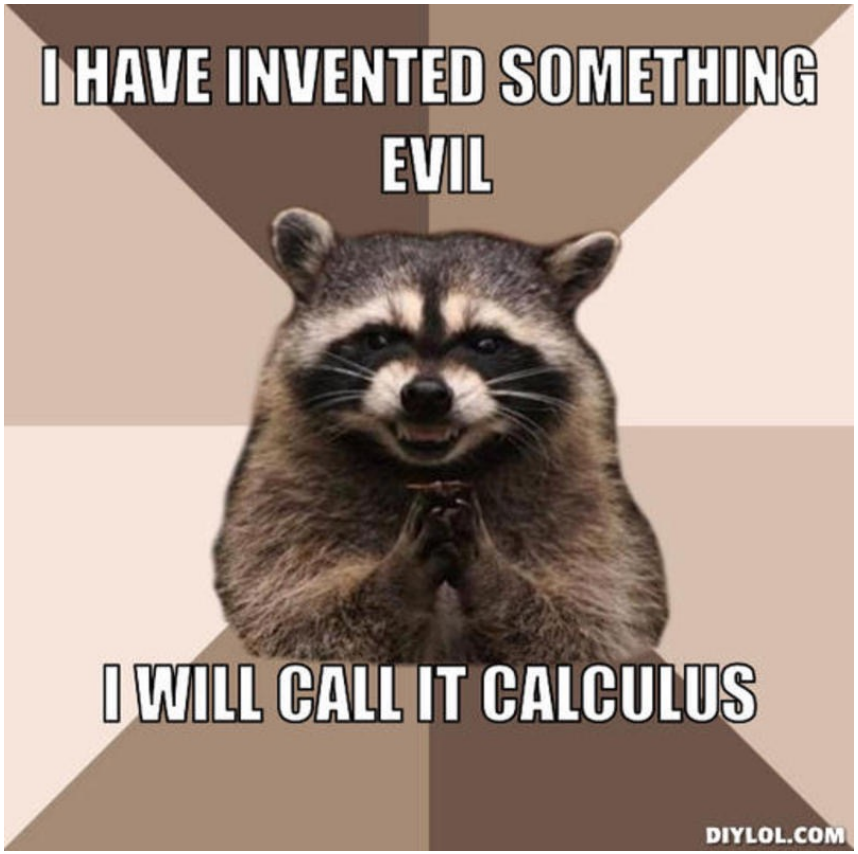
Unum evaluation refines answer to limits of the environment precision

Uboxes and solution sets

- A *ubox* is a multidimensional unum
- Exact or ULP-wide in each dimension
- Sets of uboxes constitute a *solution set*
- One dimension per degree of freedom in solution
- Solves the main problem with interval arithmetic
- Super-economical for bit storage
- Data parallel in general

Calculus considered harmful

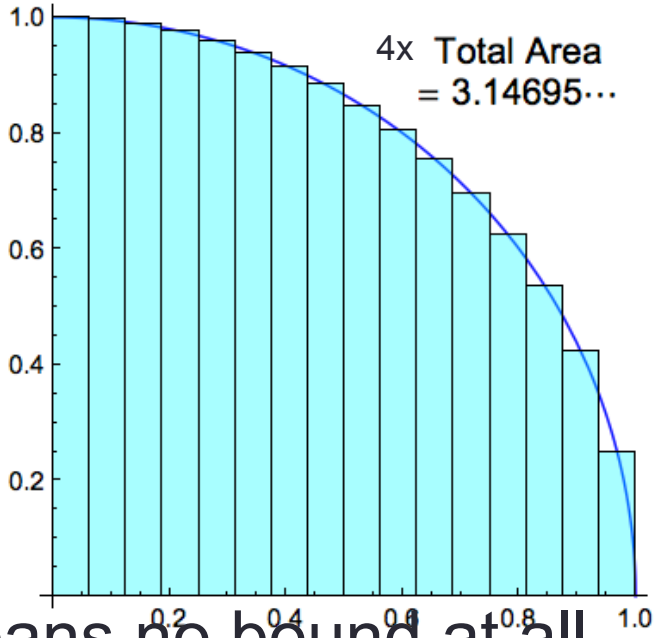
- Computers are *discrete*
- Calculus is *continuous*
- Ensures sampling errors
- Changes problem to fit the tool



Deeply Unsatisfying Error Bounds

- Classical numerical texts teach this "error bound":

$$\text{Error} \le (b - a)\, h^2\, |f''(\xi)| \,/\, 24$$



- What is f''? Where is ξ? What is the bound??
- Bound is often *infinite*, which means no bound at all
- "Dunno, but it's four times better if we make h half as big" creates demand for supercomputing that *cannot be sated*.

Two “ubox methods”, both mindless

- Paint bucket: find one solution point, test neighbors and classify as solution or fail until no more neighbors to test
  - Works if solution is known to be a connected set
  - Requires a starting point “seed”
  - Wave front of trial uboxes can be computed **in parallel**
- Try the universe: Use Warlpiri uboxes (4-bit precision) to tile all of *n*-space; increment exponent and fraction size automatically
  - 13^n things to do in parallel (!)
  - Finds *every* solution, no matter what, since all of *n*-space is represented
  - Detects ill-posed problems and solves them anyway

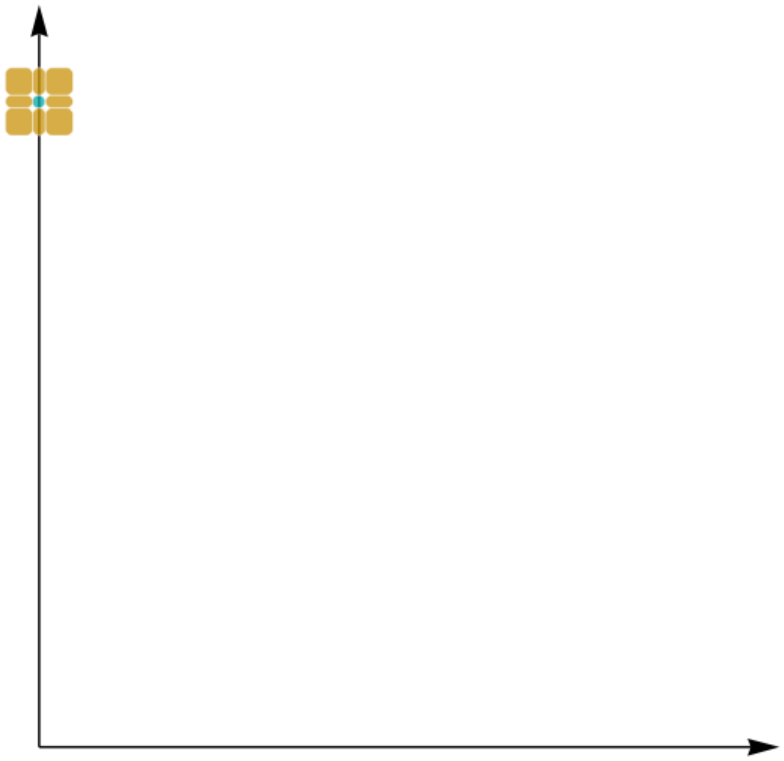
Quarter-circle example

- Suppose all we know is $x^2 + y^2 = 1$, and x and y are ≥ 0
- Suppose we have at most *2* bits exponent, *4* bits fraction

Task:
Bound the quarter circle.
Bound the value of π

Set is connected; need a seed

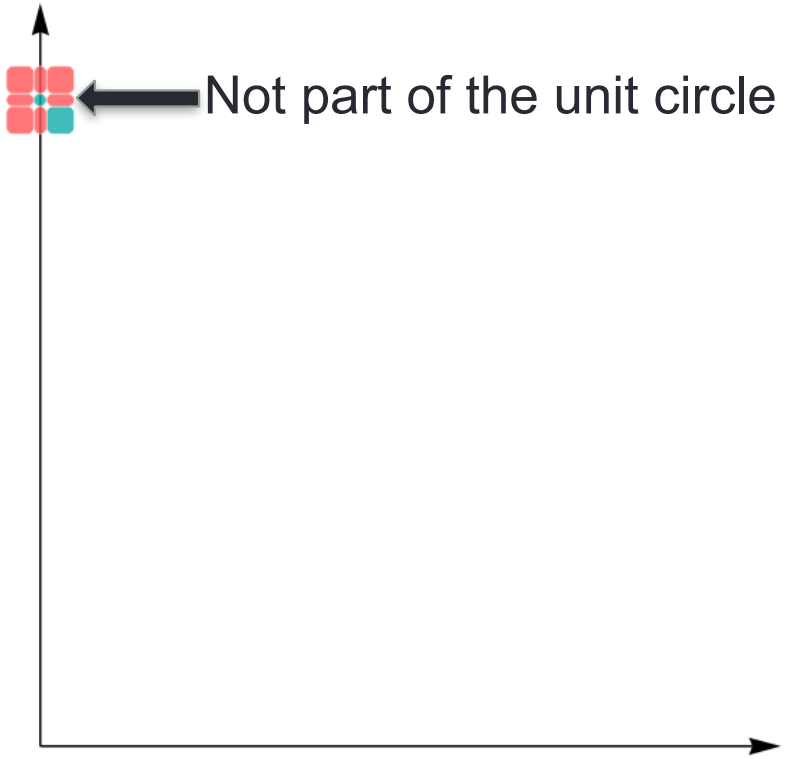
- We know $x = 0$, $y = 1$ works
- Find its eight ubox neighbors in the plane
- Test $x^2 + y^2 = 1$, $x \geq 0$, $y \geq 0$
- Solution set is green
- Trial set is amber
- Failure set is red
- Stop when no more trials



Exactly one neighbor passes

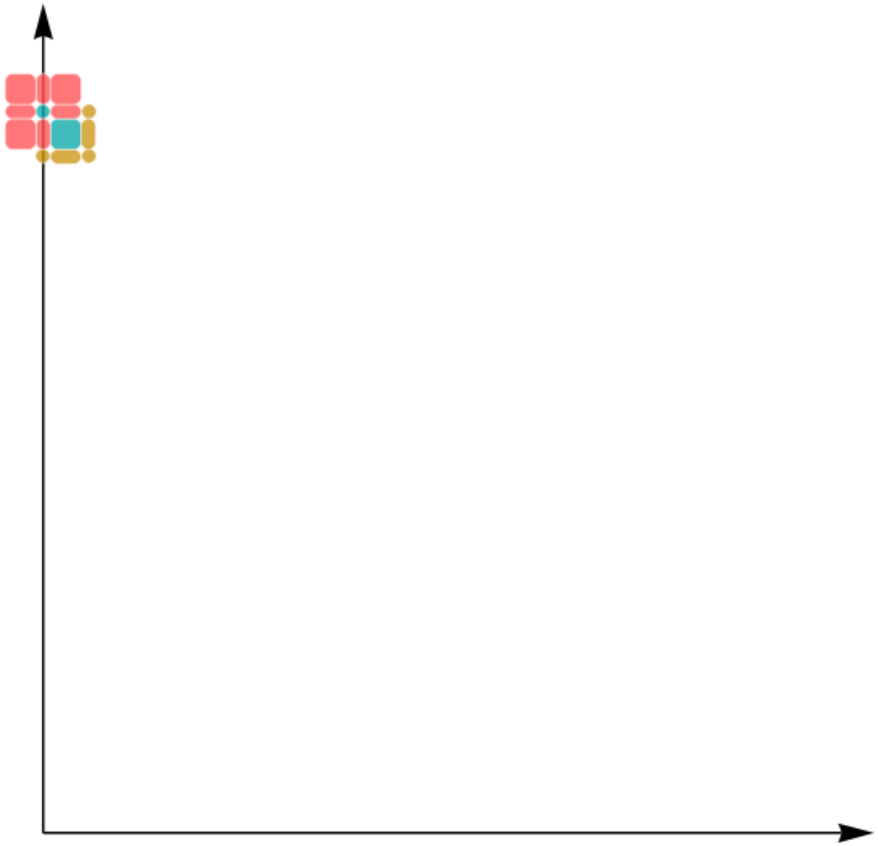
- Unum math automatically excludes cases that floats would accept
- Trials are neighbors of new solutions that
  - Are not already failures
  - Are not already solutions
- Note: *no* calculation of

$$y = \sqrt{1 - x^2}$$



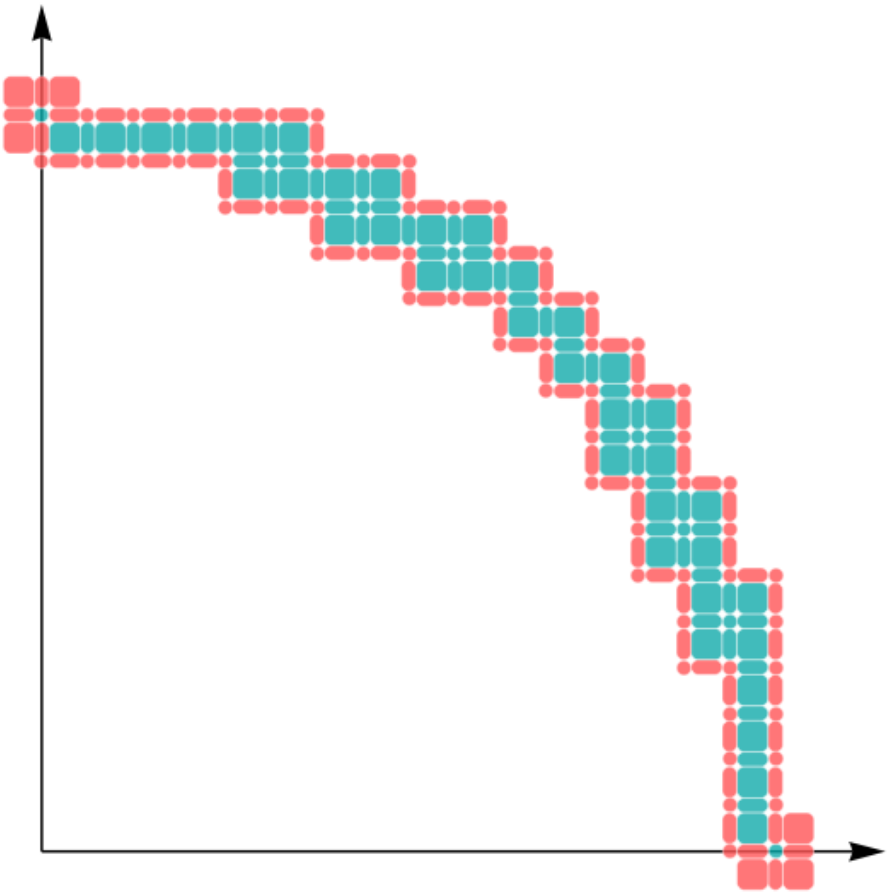
The new trial set

- Five trial uboxes to test
- Perfect, easy parallelism for multicore
- Each ubox takes only 15 to 23 bits
- Ultra-fast operations
- Cut to the chase…



The complete quarter circle

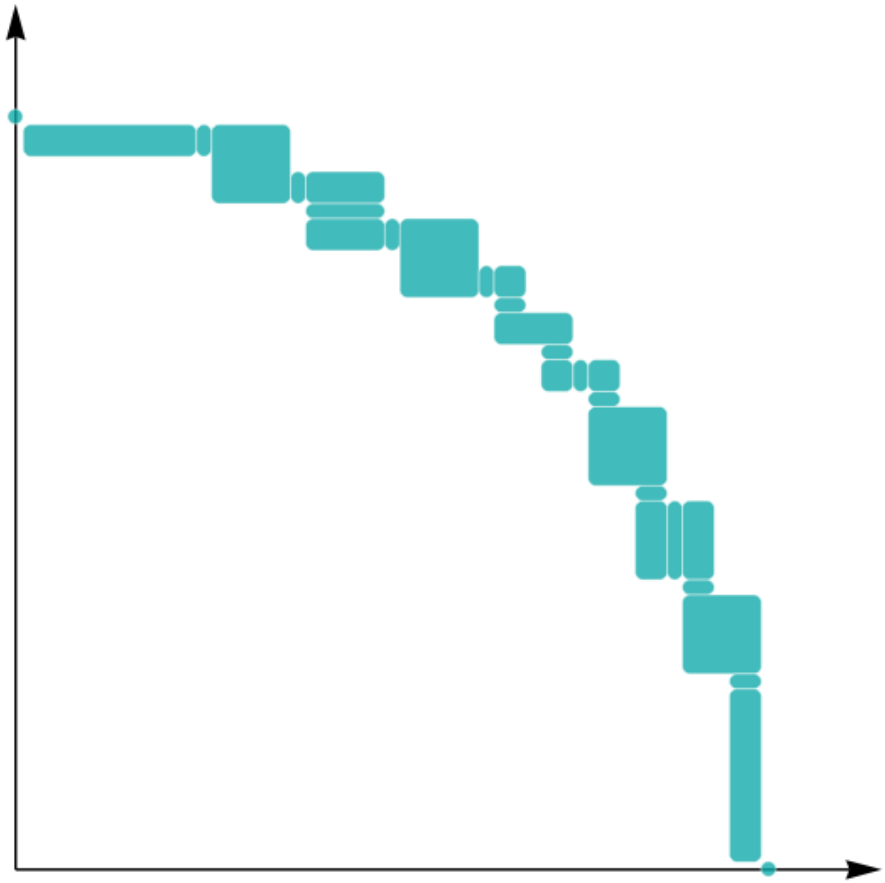
- The complete *solution*, to this finite precision level
- Circle “knowledge” is reciprocal of green area
- Use to find area under arc, bounded above and below
- *Proves* value of π to an accuracy of 3%
- No calculus needed



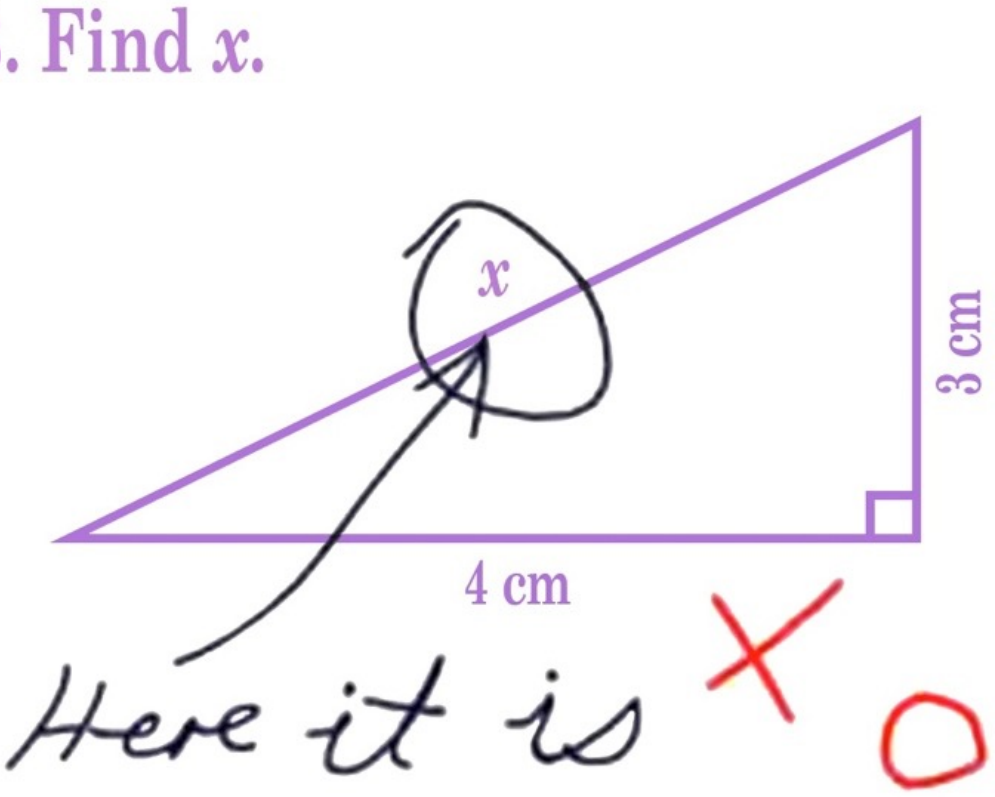
Compressed Final Result

- Coalesce uboxes to largest possible ULP values
- *Lossless* compression
- Total data set: 603 bits!

Instead of ULPs being the source of error, they are the *atomic units of computation*

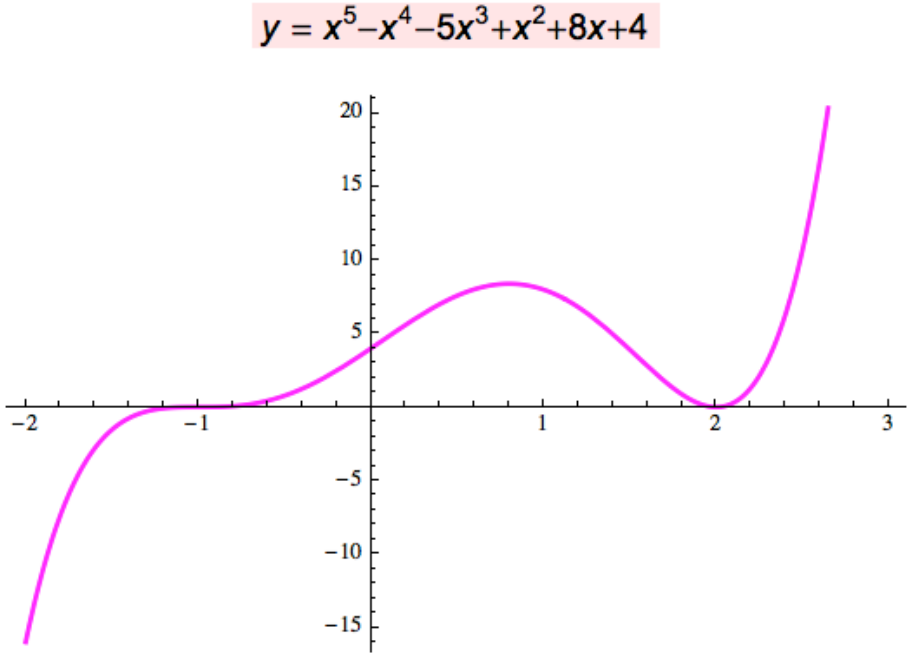


What does it mean to “solve” an equation?



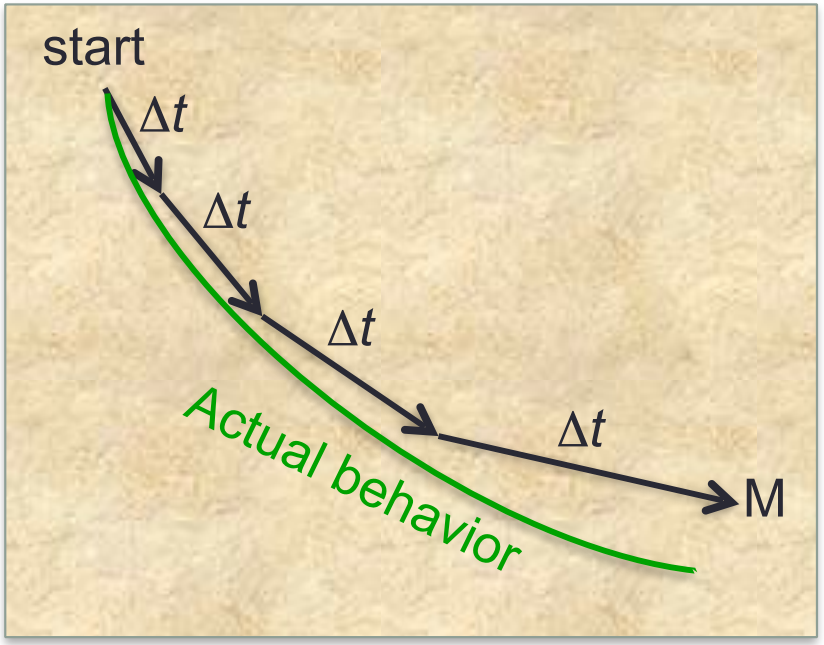
Fifth-degree polynomial roots

- Analytic solution: There isn't one.
- Numerical solution: Huge errors from rounding
- Unums: quickly return $x = -1$, $x = 2$ as the exact solutions. No rounding. No underflow. Just… the *correct answer*. With as few as 4 bits for the operands!



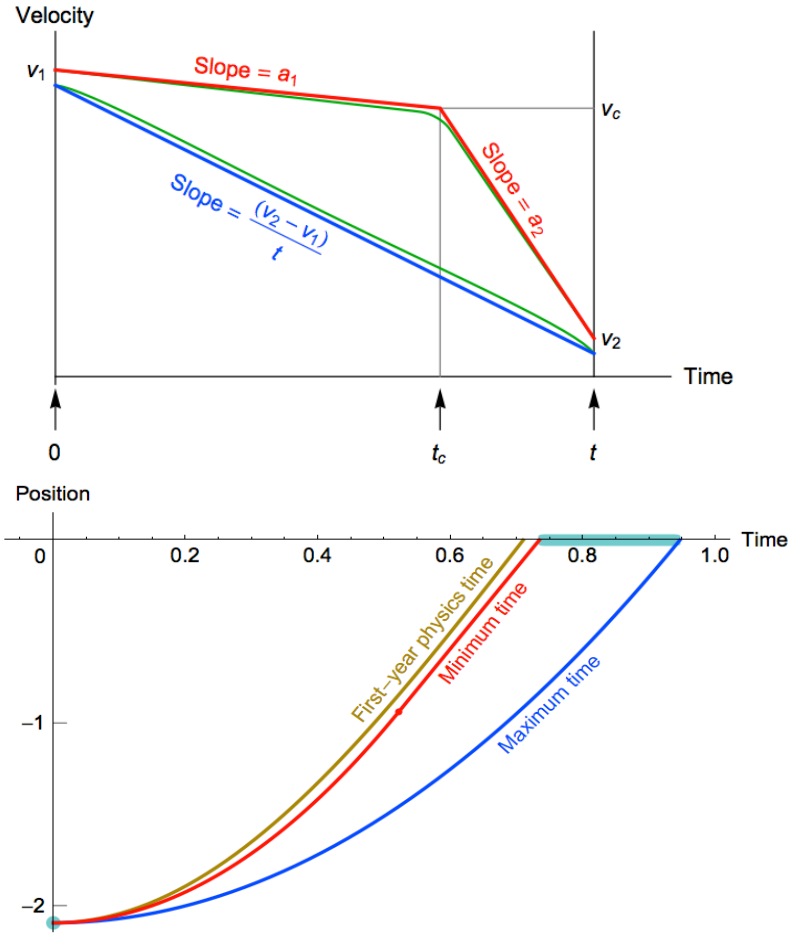
Classical Numerical Analysis

- Time steps
  - Use position to estimate force
  - Use force to estimate acceleration
  - Update the velocity
  - Update the position
  - Lather, rinse, repeat
- Accumulates rounding and sampling error, both unknown
- ***Cannot be done in parallel***



A New Type of Parallelism

- *Space* steps
- Acceleration, velocity bounded in any given space interval
- Find traversal time as a function of space step (2D ubox)
- Massively parallel!
- *No rounding error*
- *No sampling error*
- Obsoletes existing ODE methods

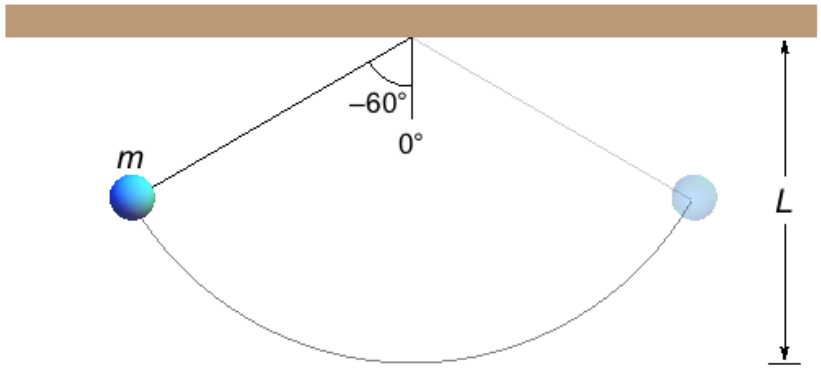


Pendulums Done Right

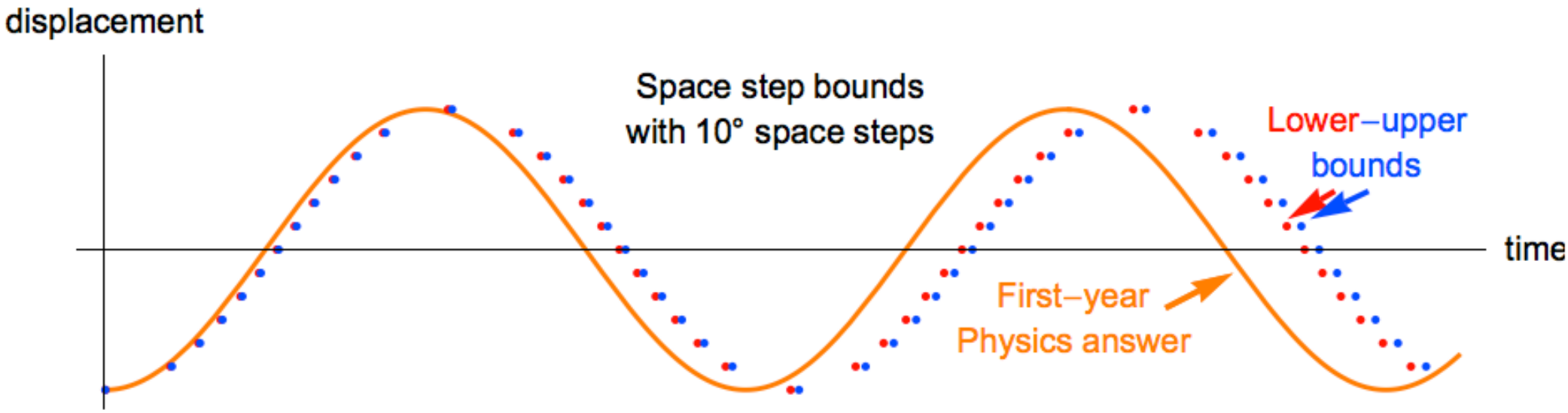
- Physics teaches us it's a harmonic oscillator with period

$$2\pi\sqrt{\frac{g}{L}}$$

- Force-fits nonlinear ODE into linear ODE for which calculus works.
- WRONG answer



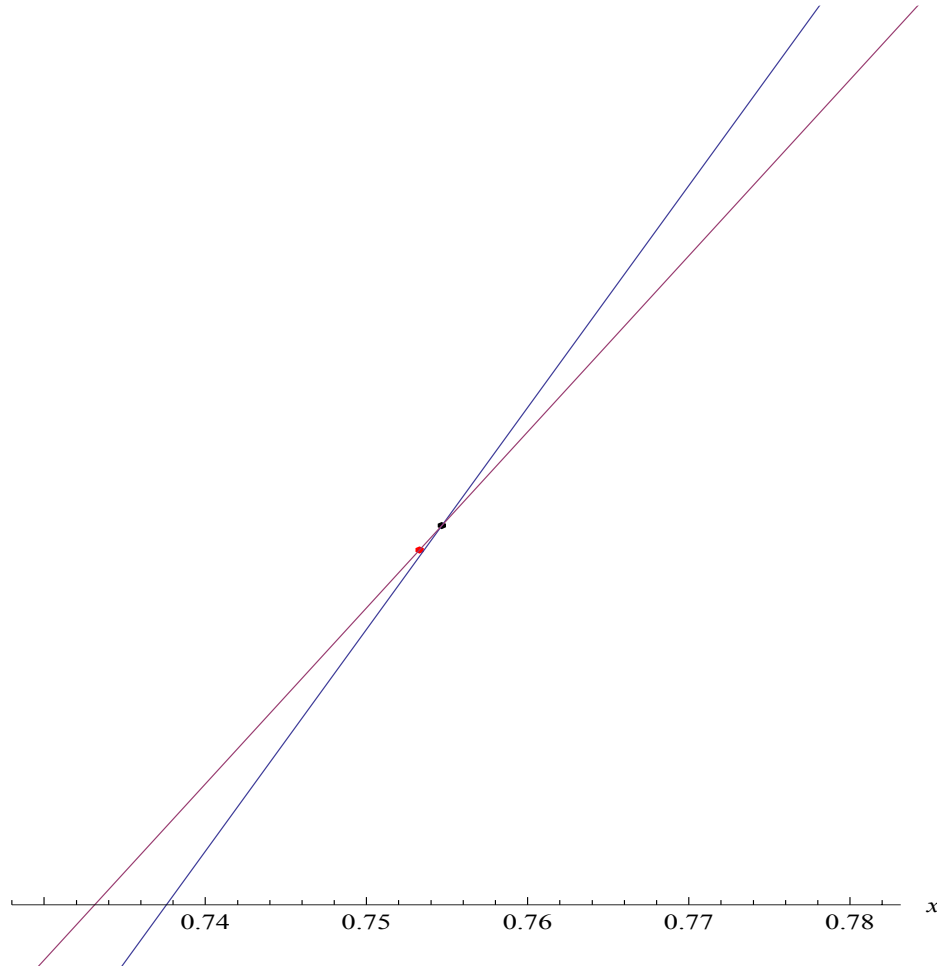
Physical Truth vs. Force-Fit Solution



You vill use calculus, and you vill LIKE IT!

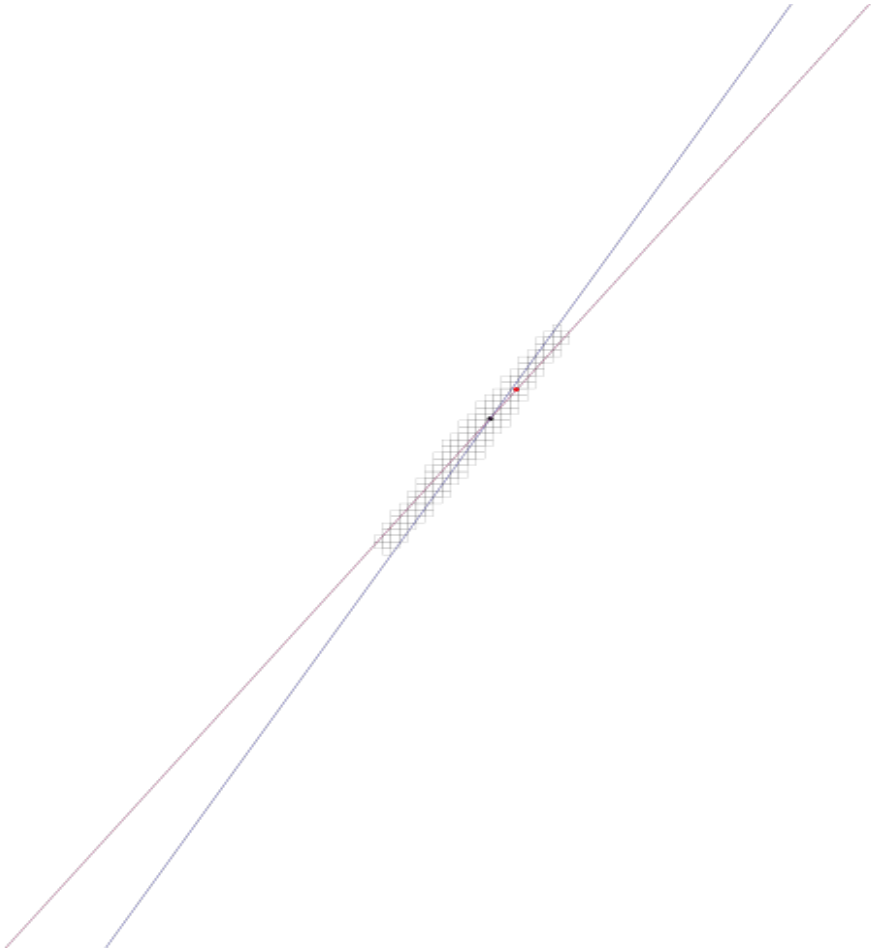
Uboxes for linear solvers

- If the *A* and *b* values in *Ax*=*b* are rounded, the “lines” have width from uncertainty
- Apply a standard solver, and get the red dot as “the answer”, *x*. A pair of floating-point numbers.
- Check it by computing *Ax* and see if it rigorously contains *b*. Yes, it does.
- Hmm… are there any *other* points that also work?



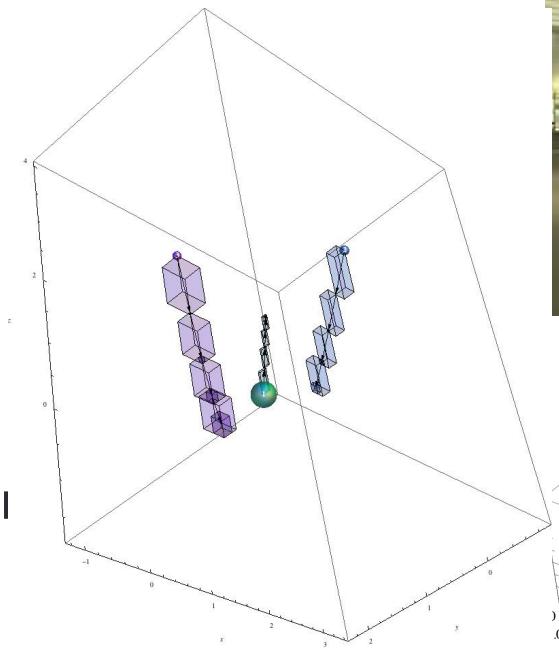
Float, Interval, and Ubox Solutions

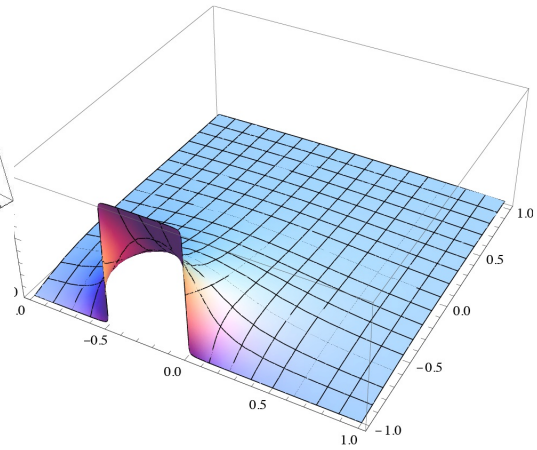
- Point solution (black dot) just gives *one of many* solutions; disguises answer instability
- Interval method (gray box) yields a bound too loose to be useful
- The ubox set (green) is the *best you can do for a given precision*
- Uboxes *reveal* ill-posed nature… yet provide solution anyway



Other Apps with Ubox Solutions

- Photorealistic computer graphics
- N-body problems
- Structural analysis
- Laplace's equation

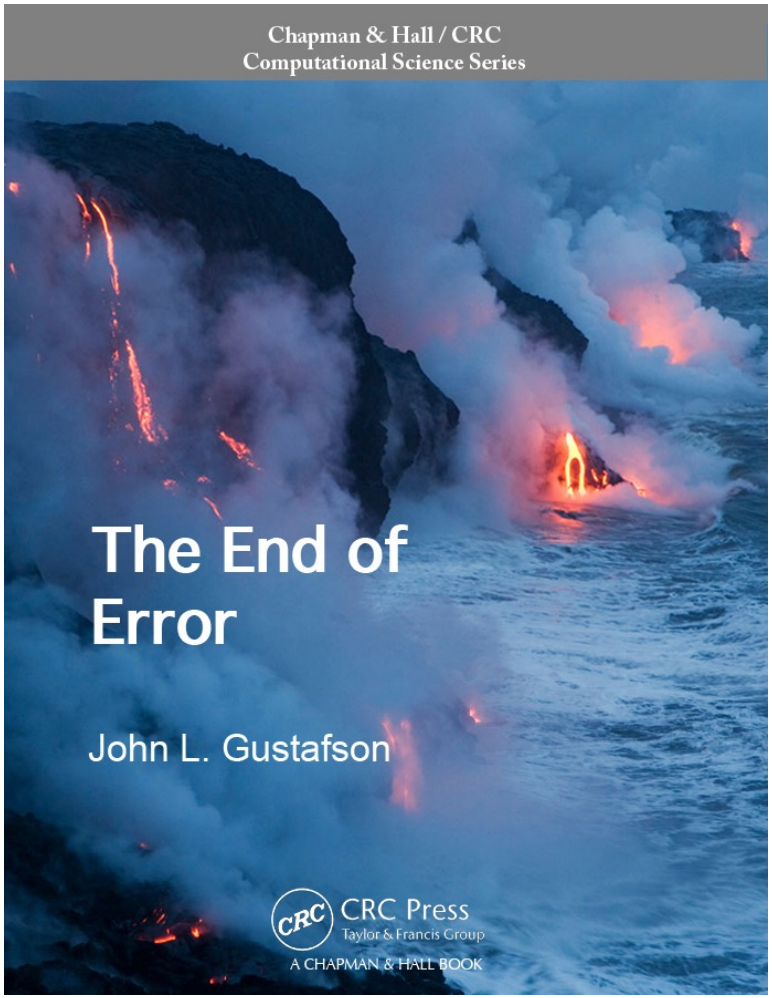




Imagine having **provable bounds** on answers for the first time, yet with easier programming, less storage, less bandwidth use, less energy/power demands, *and* abundant parallelism.

The End of Error

- A complete text on unums and uboxes will be published this year by CRC Press, with a full reference implementation
- Aimed at *general* reader; mathematicians will hate its casual style



Revisiting the Big Challenges-1

- *More hardware parallelism than we know how to use*
  - Uboxes reveal vast new sources of *data* parallelism, the easiest kind
- *Too much energy and power needed per calculation*
  - Unum arithmetic cuts the main energy hog by about 50%
- *Not enough bandwidth (the "memory wall")*
  - More use of CPU transistors, fewer bits moved to/from memory
- *Rounding errors more treacherous than people realize*
  - Unum metadata eliminates rounding error, automates precision choice
- *Rounding errors prevent use of multicore methods*
  - Unums restore algebraic laws, eliminating the parallelization deterrent

Revisiting the Big Challenges-2

- *Sampling errors turn physics simulations into guesswork*
  - Uboxes produce provable *bounds* on physical behavior
- *Numerical methods are hard to use, require expertise*
  - "Paint bucket" and "Try the universe" are brute force general methods that need no expertise
    …not even calculus

With apologies to U-Haul

For more information

Links for further information, if you are interested:

http://floating-point-gui.de/

http://en.wikipedia.org/wiki/Kahan_summation_algorithm

http://www.cs.berkeley.edu/~wl

http://ta.twi.tudelft.nl/users/vuik

http://en.wikipedia.org/wiki/IEE